

Deformed Harry Dym and Hunter-Zheng Equations

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Abstract

We study the deformed Harry Dym and Hunter-Zheng equations with two arbitrary deformation parameters. These reduce to various other known models in appropriate limits. We show that both these systems are bi-Hamiltonian with the same Hamiltonian structures. They are integrable and belong to the same hierarchy corresponding to positive and negative flows. We present the Lax pair description for both the systems and construct the conserved charges of negative order from the Lax operator. For the deformed Harry Dym equation, we construct the non-standard Lax representation for two special classes of values of the deformation parameters. In general, we argue that a non-standard description will involve a pseudo-differential operator of infinite order.

1 Introduction:

The search for exactly solvable equations has acquired enormous importance since the nonlinear Kortweg-de Vries equation was shown to be integrable [1, 2]. Exactly solvable equations, linear or nonlinear, constitute a very special class of dynamical systems with many interesting properties. The significance of determining new integrable systems, as well as studying their properties and solutions, can not be overestimated. The structure of integrable systems (or partial differential equations) is highly restrictive and, in general does not allow for deformations. In particular, the presence of arbitrary constant parameters (which can not be transformed away by some symmetry) is quite rare. One notable exception is the two boson equation [3], which has an arbitrary constant parameter present, is known to be integrable. Different values of this parameter reduce the model to other known soluble models. In this paper, we propose a new system of equations, with two arbitrary constant parameters, that is exactly soluble and reduces to various other known physical models in different limits of these parameters.

Let us recall that the Harry Dym (HD) equation [4, 5]

$$u_t = (u_{xx}^{-1/2})_x , \quad (1)$$

and the Hunter-Zheng (HZ) equation [6]

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2 , \quad (2)$$

are known to be integrable and, in fact, belong to the same hierarchy corresponding to negative and positive order flows, respectively [7]. The HD equation has interest in the study of the Saffman-Taylor problem which describes the motion of a two-dimensional interface between a viscous and a non-viscous fluid [8]. The HZ equation, on the other hand, arises in the study of massive nematic liquid crystals and in the study of shallow water waves [6]. In this paper we will show that the following system of two equations,

$$u_t = \sqrt{2} \left(\frac{1 - \lambda u_{xx}}{\sqrt{2u_{xx} - \alpha - \lambda u_{xx}^2}} \right)_x \quad (3)$$

and

$$u_{xxt} = \alpha u_x - 2u_x u_{xx} - uu_{xxx} + \frac{\lambda}{6} (u_x^3)_{xx} , \quad (4)$$

are integrable for arbitrary values of the constant parameters α, λ . Furthermore, they belong to

the same hierarchy corresponding to the negative and positive flows respectively, much like the HD and the HZ equations. It is interesting to note from (4) that when $\alpha \neq 0$, it can be scaled to unity through the scaling $x \rightarrow x/\alpha$, $u \rightarrow u/\alpha$. Therefore, looking at (4) alone, it would appear that there are only two meaningful values for α , namely, $\alpha = 0, 1$. Similarly, under a scaling $t \rightarrow t/\alpha^{3/2}$, $x \rightarrow x/\alpha$, $u \rightarrow u/\alpha$, the parameter α in the dHD equation (3) can be scaled to unity when it is not zero. However, there is no scaling which will scale α to unity (when it is nonzero) simultaneously in both the equations and, consequently, if the two systems belong to the same hierarchy, α has to be thought of as an arbitrary parameter.

We note that, for $\alpha = \lambda = 0$, eqs. (3) and (4) reduce respectively to the HD equation (1) and the HZ equation (2). For lack of a better name, we will refer to (3) as the deformed Harry Dym equation (dHD) and (4) as the deformed Hunter-Zheng equation (dHZ). (The name “generalized Harry Dym” equation has already been used earlier in the context of a multi-component Harry Dym equation [9].) For $\alpha \neq 0$ and $\lambda = 0$ equation (4) was studied by Alber et al [10] and its solutions contain solitons of the type known as umbilic solitons. Equation (4) with $\alpha = 1$ and $\lambda \neq 0$ has appeared more recently in the literature [11] as describing short capillary-gravity waves and its solutions are known to become multi-valued in a finite amount of time. While a matrix Lax pair for this system was provided in [11] and solubility of the model was argued based on a map to the sine-Gordon equation, interesting properties, such as the bi-Hamiltonian structure, the infinite number of conserved charges were not discussed at all. Our results clarify these aspects and provide direct support for the integrability of this new system. The deformed equation (3) for arbitrary α and $\lambda \neq 0$ is truly a new integrable system (which to the best of our knowledge has not been studied in the literature) and leads to the above mentioned models in different limits. This can, therefore, be considered as a very rich system, much like the two boson hierarchy.

Our paper is organized as follows. In section 2, we obtain the first Hamiltonian structure of the dHD and dHZ equations. In section 3, a second Hamiltonian structure, compatible with the first one, is obtained. We show that both the dHD and dHZ equations are bi-Hamiltonian and, therefore, integrable and belong to the same hierarchy corresponding to negative and positive flows. In section 4, we present a Lax pair description of both the system of equations and construct the conserved charges of negative order from the Lax operator. We obtain a non-standard Lax description for the dHD equation for the special values of the deformation parameters $\lambda = 0, \lambda = -3/\alpha$. We argue that, for general values of the parameters, a non-standard Lax description will involve a pseudo-differential operator of infinite order. In section 5 we present our conclusions.

2 dHZ and dHD as Hamiltonian Systems:

To describe the dHD and the dHZ equations in a compact manner, let us introduce the following notation. Let us define

$$F_{(\alpha,\lambda)}^2 \equiv 2u_{xx} - \alpha - \lambda u_{xx}^2, \quad A \equiv \frac{1 - \lambda u_{xx}}{F}. \quad (5)$$

Then, it follows from the definitions in (5) that

$$\begin{aligned} \kappa &\equiv (1 - \alpha\lambda) = \lambda F^2 + (1 - \lambda u_{xx})^2 = (\lambda + A^2)F^2, \\ \frac{F_x}{F^3} &= -\frac{1}{\kappa} A A_x, \\ \frac{u_{xxx}}{F^3} &= -\frac{1}{\kappa} A_x. \end{aligned} \quad (6)$$

These and other relations following from these prove very useful in the analysis of the two systems.

Given the dHZ equation (4), we obtain from the definition in (5) that

$$F_t = - \left[F \left(u - \frac{\lambda}{2} u_x^2 \right) \right]_x. \quad (7)$$

Similarly, in terms of the new variables in (5), the dHD equation (3) can be written as

$$u_t = \sqrt{2} A_x, \quad (8)$$

and it follows that, under the evolution of dHD,

$$\begin{aligned} F_t &= \sqrt{2} A A_{xxx}, \\ A_t &= -\sqrt{2} \kappa \frac{A_{xxx}}{F^3}. \end{aligned} \quad (9)$$

It is now straightforward to note from eqs. (7) and (9) that

$$H_{-1} = \sqrt{2} \int dx F \quad (10)$$

is conserved under both the dHD and dHZ flows.

The dHZ equation can be obtained from a variational principle, $\delta \int dt dx \mathcal{L}$, with the Lagrangian

density

$$\mathcal{L} = \frac{1}{2}u_x u_t + \frac{\alpha}{2}u^2 + \frac{1}{2}uu_x^2 - \frac{\lambda}{24}u_x^4 . \quad (11)$$

This is a first order Lagrangian density and, consequently, the Hamiltonian structure can be readily read out, or we can use, for example, Dirac's theory of constraints [12] to obtain the Hamiltonian and the Hamiltonian operator associated with (11). The Lagrangian is degenerate and the primary constraint is obtained to be

$$\Phi = \pi - \frac{1}{2}u_x , \quad (12)$$

where $\pi = \partial\mathcal{L}/\partial u_t$ is the canonical momentum. The total Hamiltonian can be written as

$$\begin{aligned} H_T &= \int dx (\pi u_t - \mathcal{L} + \beta\Phi) \\ &= \int dx \left[-\frac{\alpha}{2}u^2 - \frac{1}{2}uu_x^2 + \frac{\lambda}{24}u_x^4 + \beta \left(\pi - \frac{1}{2}u_x \right) \right] , \end{aligned} \quad (13)$$

where β is a Lagrange multiplier field. Using the canonical Poisson bracket relation

$$\{u(x), \pi(y)\} = \delta(x - y) , \quad (14)$$

with all others vanishing, it follows that the requirement of the primary constraint to be stationary under time evolution,

$$\{\Phi(x), H_T\} = 0 ,$$

determines the Lagrange multiplier field β in (13) and the system has no further constraints.

Using the canonical Poisson bracket relations (14), we can now calculate

$$K(x, y) \equiv \{\Phi(x), \Phi(y)\} = \frac{1}{2}\partial_y\delta(x - y) - \frac{1}{2}\partial_x\delta(x - y) . \quad (15)$$

This shows that the constraint (12) is second class and that the Dirac bracket between the basic variables has the form

$$\{u(x), u(y)\}_D = \{u(x), u(y)\} - \int dz dz' \{u(x), \Phi(z)\} J(z, z') \{\Phi(z'), u(y)\} = J(x, y) ,$$

where J is the inverse of the Poisson bracket of the constraint (15),

$$\int dz K(x, z) J(z, y) = \delta(x - y) .$$

This last relation determines

$$\partial_x J(x, y) = \delta(x - y) ,$$

or

$$J(x, y) = \mathcal{D}_1 \delta(x - y) ,$$

where

$$\mathcal{D}_1 = \partial^{-1} , \tag{16}$$

and can be thought of as the alternating step function in the coordinate space. We can now set the constraint (12) strongly to zero in (13) to obtain

$$H_2 \equiv -H_T = \int dx \left(\frac{\alpha}{2} u^2 + \frac{1}{2} u u_x^2 - \frac{\lambda}{24} u_x^4 \right) . \tag{17}$$

Therefore, the dHZ equation can be written in the Hamiltonian form

$$u_t = \mathcal{D}_1 \frac{\delta H_2}{\delta u} ,$$

with \mathcal{D}_1 and H_2 given by (16) and (17), respectively.

From the results in [7] we know that the HD and HZ equations belong to the same hierarchy of equations. Here, too, we will see that both the dHD and the dHZ equations belong to the same hierarchy. In particular, we note that

$$u_t = \mathcal{D}_1 \frac{\delta H_{-1}}{\delta u} ,$$

with H_{-1} , given by (10), yields the deformed Harry Dym equation (8). As a result, the dHD equation also is Hamiltonian with the same Hamiltonian structure of the dHZ equation in (16) and, consequently, has a Lagrangian description given by

$$\mathcal{L} = \frac{1}{2} u_x u_t - \sqrt{2} F .$$

The reader can easily check that H_2 is also conserved by both the dHD and dHZ equations, much like H_{-1} . Note that for $\alpha = \lambda = 0$, these two charges reduce to the corresponding ones in the HD and HZ systems.

3 dHD and dHZ as bi-Hamiltonian Systems:

It is well known that a system can be shown to be integrable if it is bi-Hamiltonian [13, 14]. This corresponds to the system having a Hamiltonian description with two distinct Hamiltonian structures that are compatible. Therefore, we try to see if the dHD and the dHZ equations can be described as bi-Hamiltonian systems. For this, we have to find a second Hamiltonian description for the two systems.

In order to determine a second Hamiltonian structure for the two systems, we recall [7] that, when $\alpha = \lambda = 0$, the corresponding H_{-1} is a Casimir of the second Hamiltonian structure for the HD and HZ systems, namely,

$$\mathcal{D}_2^{(\alpha=\lambda=0)} = \partial^{-2} u_{xx} \partial^{-1} + \partial^{-1} u_{xx} \partial^{-2} \quad (18)$$

satisfies

$$\mathcal{D}_2^{(\alpha=\lambda=0)} \frac{\delta H_{-1}^{(\alpha=\lambda=0)}}{\delta u} = 0 .$$

Since we know H_{-1} for the deformed systems which is a generalization of $H_{-1}^{(\alpha=\lambda=0)}$, we look for a Hamiltonian structure for which it is a Casimir. With some work, it can be determined that H_{-1} given in (10) is a Casimir of

$$\mathcal{D}_2 \equiv \mathcal{D}_2^{(\alpha,\lambda)} = \frac{1}{2} (\partial^{-2} F^2 \partial^{-1} + \partial^{-1} F^2 \partial^{-2}) + \lambda \partial^{-2} u_{xxx} \partial^{-1} u_{xxx} \partial^{-2} . \quad (19)$$

Note that this structure reduces to (18) when $\alpha = \lambda = 0$. The skew symmetry of this Hamiltonian structure is manifest. The proof of the Jacobi identity for this structure as well as its compatibility with \mathcal{D}_1 in (16) can be determined through the standard method of prolongation described in ref. [14], which we discuss briefly.

Performing the change of variables

$$w = u_{xx} ,$$

the Hamiltonian structures (16) and (19) assume the forms

$$\begin{aligned} \mathcal{D}_1 &= \partial^3 , \\ \mathcal{D}_2 &= \frac{1}{2} (F^2 \partial + \partial F^2) + \lambda w_x \partial^{-1} w_x . \end{aligned}$$

We can construct the two bivectors associated with the two structures as

$$\begin{aligned}\Theta_{\mathcal{D}_1} &= \frac{1}{2} \int dx \{ \theta \wedge \mathcal{D}_1 \theta \} = \frac{1}{2} \int dx \theta \wedge \theta_{xxx} , \\ \Theta_{\mathcal{D}_2} &= \frac{1}{2} \int dx \{ \theta \wedge \mathcal{D}_2 \theta \} \\ &= \frac{1}{2} \int dx \{ -\alpha \theta \wedge \theta_x + 2w \theta \wedge \theta_x - \lambda w^2 \theta \wedge \theta_x + \lambda w_x \theta \wedge (\partial^{-1} w_x \theta) \} .\end{aligned}$$

Using the prolongation relations,

$$\begin{aligned}\mathbf{pr} \vec{v}_{\mathcal{D}_1 \theta}(w) &= \theta_{xxx} , \\ \mathbf{pr} \vec{v}_{\mathcal{D}_1 \theta}(w^2) &= 2w \mathbf{pr} \vec{v}_{\mathcal{D}_1 \theta}(w) , \\ \mathbf{pr} \vec{v}_{\mathcal{D}_1 \theta}(w_x) &= (\mathbf{pr} \vec{v}_{\mathcal{D}_1 \theta}(w))_x , \\ \mathbf{pr} \vec{v}_{\mathcal{D}_2 \theta}(w) &= -\alpha \theta_x + 2w \theta_x - \lambda w^2 \theta_x + w_x \theta - \lambda w w_x \theta + \lambda w_x (\partial^{-1} w_x \theta) , \\ \mathbf{pr} \vec{v}_{\mathcal{D}_2 \theta}(w^2) &= 2w \mathbf{pr} \vec{v}_{\mathcal{D}_2 \theta}(w) , \\ \mathbf{pr} \vec{v}_{\mathcal{D}_2 \theta}(w_x) &= (\mathbf{pr} \vec{v}_{\mathcal{D}_2 \theta}(w))_x ,\end{aligned}\tag{20}$$

it is straightforward to show that the prolongation of the bivector $\Theta_{\mathcal{D}_2}$ vanishes,

$$\mathbf{pr} \vec{v}_{\mathcal{D}_2 \theta}(\Theta_{\mathcal{D}_2}) = 0 ,$$

implying that \mathcal{D}_2 satisfies Jacobi identity. Using (20), it also follows that

$$\mathbf{pr} \vec{v}_{\mathcal{D}_1 \theta}(\Theta_{\mathcal{D}_2}) + \mathbf{pr} \vec{v}_{\mathcal{D}_2 \theta}(\Theta_{\mathcal{D}_1}) = 0 ,$$

showing that \mathcal{D}_1 and \mathcal{D}_2 are compatible. Namely, not only are $\mathcal{D}_1, \mathcal{D}_2$ genuine Hamiltonian structures, any arbitrary linear combination of them is as well. Any physical system that is Hamiltonian with respect to these two structures, therefore, defines a pencil system and is integrable. It is worth pointing out here that when $\alpha = \lambda = 0$, the second Hamiltonian structure (18) represents the centerless Virasoro algebra [15] (with dimension zero operators). The structure in (19) appears to be a highly nonlocal generalization of this algebra, but we are not familiar with any study of such an algebra in the literature.

To show that the dHD and dHZ are bi-Hamiltonian systems, we note that the charges

$$H_1 = \int dx u_x^2 ,\tag{21}$$

$$H_{-2} = \frac{1}{2\kappa} \int dx F A_x^2 \quad (22)$$

are also conserved by both the dHZ and dHD equations. While the charge in (21) is the unmodified charge of the HD and HZ systems (this seems to be a simple coincidence), the charge in (22) is a true generalization of the corresponding charge of the HD and HZ systems. With these charges, it is easy to see that the dHZ equation can be written in a truly bi-Hamiltonian form

$$u_t = \mathcal{D}_1 \frac{\delta H_2}{\delta u} = \mathcal{D}_2 \frac{\delta H_1}{\delta u} .$$

Similarly, the dHD equation can also be written in the bi-Hamiltonian form

$$u_t = \mathcal{D}_1 \frac{\delta H_{-1}}{\delta u} = \mathcal{D}_2 \frac{\delta H_{-2}}{\delta u} .$$

Thus, we see that both the dHD as well as dHZ systems are bi-Hamiltonian with the same two compatible Hamiltonian structures and are, therefore, integrable.

4 The Lax Representation:

When a system is bi-Hamiltonian, we can naturally define a hierarchy of commuting flows through the relation

$$u_t = K_n[u] = \mathcal{D}_1 \frac{\delta H_{n+1}}{\delta u} = \mathcal{D}_2 \frac{\delta H_n}{\delta u} , \quad n = 0, 1, 2, \dots \quad (23)$$

In the present case, both the Hamiltonian structures have Casimirs. We have already seen that H_{-1} is a Casimir of \mathcal{D}_2 and it can be checked that the trivial Hamiltonian

$$H_0 = \int dx u_{xx} \quad (24)$$

formally defines the Casimir of \mathcal{D}_1 (namely, if we write formally $\frac{\delta H_0}{\delta u} = \partial^2$, it is annihilated by \mathcal{D}_1). As a result, the system of flows can be extended to both positive and negative integer values for n . In this way, we see that much like in the HD and HZ systems [7], the dHD and dHZ systems also belong to the same hierarchy corresponding to the negative and positive flows.

Let us introduce the recursion operator following from the two Hamiltonian structures as

$$R = \mathcal{D}_2 \mathcal{D}_1^{-1} .$$

Then, it follows from (23) that

$$K_{n+1} = R K_n ,$$

and

$$\frac{\delta H_{n+1}}{\delta u} = R^\dagger \frac{\delta H_n}{\delta u} , \quad (25)$$

where

$$R^\dagger = \partial^{-1} u_{xx} \partial^{-1} + (-\alpha + u_{xx}) \partial^{-2} - \lambda u_{xx} \partial^{-1} u_{xx} \partial^{-1} \quad (26)$$

is the adjoint of R . The conserved charges for the hierarchy can, of course, be determined in principle recursively from (25). However, in practice, integrating the recursion relation is highly nontrivial. Therefore, we look for a Lax representation for the system of dHD and dHZ equations which will allow us to construct the conserved charges directly.

It is well known [16, 17] that for a bi-Hamiltonian system of evolution equations, $u_t = K_n[u]$, a natural Lax description

$$\frac{\partial M}{\partial t} = [M, B] ,$$

is easily obtained where, we can identify

$$M \equiv R^\dagger ,$$

$$B \equiv K'_n .$$

Here K'_n represents the Fréchet derivative of K_n , defined by

$$K'_n[u] v = \frac{d}{d\epsilon} K_n[u + \epsilon v] \Big|_{\epsilon=0} .$$

For the dHD and dHZ system of equations in (3) and (4) respectively, we have

$$B^{\text{dHD}} \equiv K_{-2} = \sqrt{2} \kappa \partial^2 F^{-3} \partial ,$$

$$B^{\text{dHZ}} \equiv K_1 = (-\alpha + u_{xx}) \partial^{-1} + \partial \left(u - \frac{\lambda}{2} u_x^2 \right) .$$

The two systems have the same $M = R^\dagger$ given in (26). It can now be checked that

$$\begin{aligned}\frac{\partial M}{\partial t} &= [M, B^{\text{dHD}}] , \\ \frac{\partial M}{\partial t} &= [M, B^{\text{dHZ}}] ,\end{aligned}\tag{27}$$

do indeed generate the dHD and the dHZ equations and, thereby, provide a Lax pair for the system.

One of the advantages of a Lax representation is that they directly give the conserved charges of the system. From the structure of (27), it follows that $\text{Tr} M^{\frac{2n+1}{2}}$ are conserved, where “Tr” represents Adler’s trace [18]. We note that

$$\begin{aligned}\text{Tr} M^{\frac{2n+1}{2}} &= 0 , \quad n \geq 1 \\ \text{Tr} M^{\frac{1}{2}} &= \int dx F , \\ \text{Tr} M^{-\frac{1}{2}} &= -\frac{1}{2\kappa} \int dx F A_x^2 , \\ \text{Tr} M^{-\frac{3}{2}} &= \frac{3}{\kappa} \int dx \left(4 \frac{A_{xx}^2}{F} + \frac{1}{\kappa} F A_x^4 \right) , \\ &\vdots\end{aligned}\tag{28}$$

The first two nontrivial charges correspond respectively to H_{-1}, H_{-2} given in eqs. (10) and (22), constructed earlier by brute force. In fact, all H_{-n-1} with positive $n \geq 0$ can be constructed from $\text{Tr} M^{-\frac{2n+1}{2}}$ and by construction (namely, because of the nature of (27)), they are conserved under both the dHD and dHZ flows. Unfortunately, as is clear from (28), this procedure does not yield the charges H_n with positive integer values. These are, in general, non-local and even in the HD and HZ case, construction of these charges relies primarily on the recursion relation (25). It remains an interesting question to construct these charges in a more direct manner.

The Harry Dym equation has a Gelfand-Dikii representation for the Lax pair, while the HZ equation does not. We will now discuss the existence of such a Lax representation for the dHD equation. A spectral problem associated with the dHD equations can be obtained from the recursion relation (25) (see [19] and references therein) as follows. Introducing a spectral parameter μ and defining

$$\psi^2(x, t, \mu) = \sum_{n=0}^{\infty} \mu^n \frac{\delta H_n}{\delta u} ,$$

we note that the recursion relation (25) can be written compactly as (recall that H_0 is a Casimir of \mathcal{D}_1)

$$(\mathcal{D}_1 - \mu \mathcal{D}_2) \psi^2 = 0 ,$$

or

$$(1 - \mu R^\dagger) \psi^2 = 0 , \tag{29}$$

which defines an eigenvalue problem for the eigenfunction ψ^2 with eigenvalue $1/\mu$. A linear eigenvalue problem can be derived from this if we can factorize the operator $(1 - \mu R^\dagger)$.

Let us note that the operator R^\dagger in (26) can be rewritten in the form

$$R^\dagger = \frac{1}{2} \left[\partial^{-1} \left(F_{(\alpha, \lambda)}^2 + X \right) \partial^{-1} + \left(F_{(\alpha, \lambda)}^2 + X \right) \partial^{-2} \right] , \tag{30}$$

where

$$X = \sum_{n=1} X_n \partial^{-n} ,$$

and the coefficients X_n can be determined recursively to be

$$\begin{aligned} X_1 &= 0 , \\ X_2 &= \frac{\lambda}{4} u_{xxx}^2 , \\ X_3 &= -\frac{1}{2} X_{2,x} , \\ &\vdots \quad . \end{aligned}$$

When $\lambda = 0$, it follows that $X = 0$ and that $F_{(\alpha, \lambda=0)}^2$ is a simple function. In this case, the eigenvalue problem (29) can be factorized as

$$(1 - R^\dagger) \psi^2 = 2 \partial^{-1} \phi^{-2} \partial \phi^3 \left(\partial^2 - \frac{\mu}{4} F_{(\alpha, \lambda=0)}^2 \right) \phi = 0 ,$$

where we have identified

$$\phi^2 = (\partial^{-2} \psi^2) .$$

This shows that if the linear equation

$$\left(\partial^2 - \frac{\mu}{4} F_{(\alpha, \lambda=0)}^2\right) \phi = 0$$

is satisfied, then (29) will hold and this identifies the Lax operator for the system to be

$$L = \frac{1}{F_{(\alpha, \lambda=0)}^2} \partial^2 .$$

In fact, it can be readily checked that when $\lambda = 0$, the hierarchy of dHD equations can be obtained from the non-standard Lax equation

$$\frac{\partial L}{\partial t_n} = 4\sqrt{2} \left[L, (L^{(2n-1)/2})_{\geq 2} \right] .$$

The conserved quantities for this system can be obtained from $\text{Tr } L^{(2n-1)/2}$, $n = 0, 1, 2, \dots$

On the other hand, when $\lambda \neq 0$, the coefficients X_n are nontrivial and X represents a pseudo-differential operator. The factorization, in such a case, is not so simple and, in principle would involve an infinite series of terms. For arbitrary values of κ , the terms in the series can possibly be determined recursively. However, this is not very interesting. We simply note here that for the special value $\kappa = 4$, the infinite series of terms seems to have a simpler compact form and the Lax operator, in such a case, has the form

$$L_{\kappa=4} = \frac{1}{F} \partial \frac{1}{F} \partial - \frac{1}{4} A_x \partial^{-1} A_x \partial ,$$

and, in this case, the hierarchy of dHD equations can be obtained from the non-standard Lax representation

$$\frac{\partial L_{\kappa=4}}{\partial t_n} = 4\sqrt{2} \left[L_{\kappa=4}, \left(L_{\kappa=4}^{(2n-1)/2} \right)_{\geq 2} \right] .$$

The conserved quantities, in this case, also follow from $\text{Tr } L_{\kappa=4}^{(2n-1)/2}$ and up to multiplicative factors, they have the forms given in (28) with $\kappa = 4$. A simple Lax description for arbitrary λ , however, remains an open question.

5 Conclusion:

In this paper, we have studied the general system of dHD and dHZ equations. We have shown that both these systems are bi-Hamiltonian and, therefore, integrable and belong to the same hierarchy

corresponding to negative and positive flows. The Lax pair for the two system of equations have been derived and conserved charges corresponding to negative integer values follow from the Lax operator. A simple construction of the charges for positive integer values remains an open question. For $\lambda = 0$, we have constructed a non-standard Lax representation for the dHD equation, which involves a purely differential Lax operator. For arbitrary values of λ , we have argued that a non-standard Lax representation will necessarily involve a Lax operator which is a pseudo-differential operator of infinite order. For the particular case of $\kappa = 4$, however, this takes a simpler compact form.

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